A general solution of the equations of hydrodynamics

By L. M. MILNE-THOMSON
Brown University, Providence, Rhode Island

(*Received* 9 *October* 1956)

Consider a fluid of constant viscosity μ. If the fluid is incompressible, we suppose it to move in a conservative field in which the force per unit mass is $-\nabla\Omega$. If the fluid is compressible, we suppose the field of force to be absent. Both cases are included if we write $V = \epsilon\rho\Omega$, where $\epsilon = 1$ for an incompressible fluid, and $\epsilon = 0$ for a compressible fluid. The equation of continuity

$$\partial\rho/\partial t + \nabla.(\rho\mathbf{q}) = 0 \qquad (1)$$

can be satisfied identically by

$$\rho = \nabla^2\chi, \qquad \rho\mathbf{q} = -\nabla(\partial\chi/\partial t), \qquad (2)$$

where χ is an arbitrary (differentiable) scalar function of position.

The equation of motion is (Milne-Thomson 1955, §§ 19.02, 19.03)

$$\rho\, d\mathbf{q}/dt = -\nabla V + \nabla.\Phi, \qquad (3)$$

where the stress tensor is

$$\Phi = -(p + \tfrac{2}{3}\mu\nabla.\mathbf{q})I + \mu(\nabla;\mathbf{q} + \mathbf{q};\nabla), \qquad (4)$$

and I is the idemfactor or unit tensor of the second rank.

Now (3) can be written

$$\rho\, d\mathbf{q}/dt = \nabla.(\Phi - VI). \qquad (5)$$

Using (1) we find that (Milne-Thomson 1955, § 2.34)

$$\rho\, d\mathbf{q}/dt = \partial(\rho\mathbf{q})/\partial t + \nabla.(\rho\mathbf{q};\mathbf{q}) = \nabla.(-I\partial^2\chi/\partial t^2 + \rho\mathbf{q};\mathbf{q})$$

and therefore (5) becomes

$$\nabla.\{\Phi + I(\partial^2\chi/\partial t^2 - V) - \rho\mathbf{q};\mathbf{q}\} = 0,$$

which can be satisfied identically by

$$\Phi = I(V - \partial^2\chi/\partial t^2) + \rho\mathbf{q};\mathbf{q} + \nabla\wedge\Psi\wedge\nabla, \qquad (6)$$

wherein Ψ is an arbitrary tensor of the second rank (Milne-Thomson 1942; that $\nabla.(\nabla\wedge\Psi\wedge\nabla) = 0$ is a consequence of Theorem I of this paper). Equations (2) and (6) furnish the density, velocity, and stress distribution in terms of an arbitrary function and an arbitrary tensor of the second rank.

References

Milne-Thomson, L. M. 1942 Consistency equations for the stresses in isotropic elastic and plastic materials. *J. London Math. Soc.* **17**, 115–128.

Milne-Thomson, L. M. 1955 *Theoretical Hydrodynamics*, 3rd Ed. London: Macmillan.